MATHEMATICS METHODS

MAWA Semester 1 (Unit 3) Examination 2020
Calculator-assumed

Marking Key

Section One: Calculator-assumed (100 Marks)

Question 8(a) (3 marks)

Solution	
$c = Ae^{-kt}$ $t = 0, c = 0.03 \Rightarrow A = 0.03.$ $t = 36.5, c = \frac{A}{2} \Rightarrow \frac{A}{2}e^{-k\times36.5} \Rightarrow 0.5 = e^{-k\times36.5} \Rightarrow k \approx 0.019.$	
Mathematical behaviours	**Marks**
• uses initial condition to construct equation and solves for A • constructs equation related to half life • solves for k	1 1 1

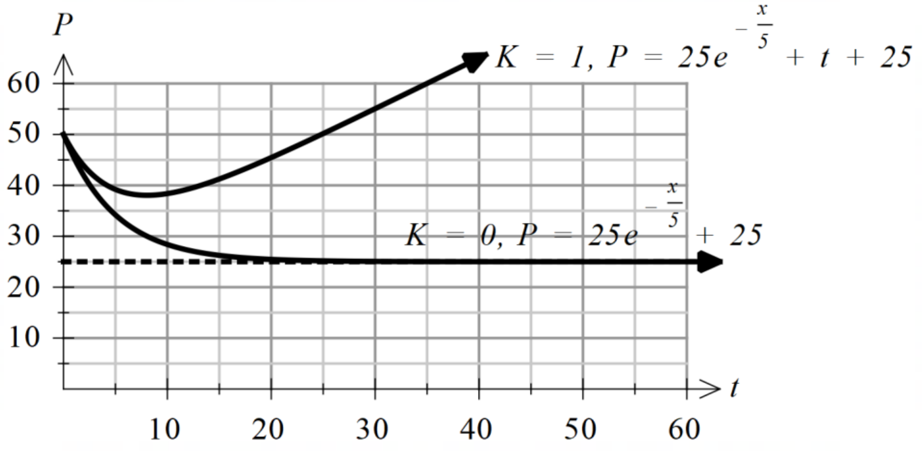
Question 8(b) (2 marks)

Solution	
For isotope B, $A = 0.02$. $0.5 = e^{-k\times62.9} \Rightarrow k \approx 0.011.$ Solving $0.02e^{-0.011} = 2\times0.03e^{-0.019t} \Rightarrow t \approx 137.3$ Hence approximately 137 years from now.	
Mathematical behaviours	**Marks**
• states equation to be solved • solves for t and states time (in years)	1 1

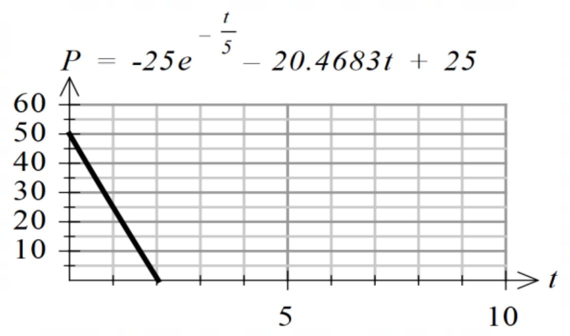
Question 9(a) **(2 marks)**

Solution	
$\frac{dP}{dt} = -5e^{-\frac{t}{5}} + K \Rightarrow P = 25e^{-\frac{t}{5}} + Kt + d$ $t = 0, P = 50 \Rightarrow 50 = 25e^{0} + d \Rightarrow d = 25$ $\therefore P = 25e^{-\frac{t}{5}} + Kt + 25$ as required.	
Mathematical behaviours	**Marks**
• anti-differentiates the derivative function correctly • uses the initial condition to find the constant of integration and deduces the required solution	1 1

Question 9(b) **(4 marks)**

Solution	
(i) $K = 1, P = 25e^{-\frac{t}{5}} + t + 25$ As $t \to \infty, P \to \infty$ i.e. population of infected people increases indefinitely (ii) $K = 0, P = 25e^{-\frac{t}{5}} + t(0) + 25 = 25e^{-\frac{t}{5}} + 25$ As $t \to \infty, P \to 25$ i.e. population of infected people stabilises to 25 (iii) 	
Mathematical behaviours	**Marks**
(i) • recognises that when K = 1, and as $t \to \infty, P \to \infty$ ie population of infected people increases indefinitely	1
(ii) • recognises that when K = 0 and as $t \to \infty, P \to 25$ ie population of infected people stabilises to 25	1
(iii) • correct graph for K= 0 • correct graph for K = 1	1 1

Question 9(c) **(3 marks)**

Solution	
$P = 25e^{-\frac{t}{5}} + Kt + 25$ $t = 1, P = 25 \Rightarrow 25 = 25e^{-\frac{1}{5}} + K + 25$ $K = -25e^{-\frac{1}{5}} \approx -20.4683$ Hence, $P = 25e^{-\frac{t}{5}} - 20.4683t + 25$ From CAS, this is a decreasing function. $P = 0 \Rightarrow t = 2.0345$ ie Population of infected people will reduce to zero after 2.0345 weeks	

Mathematical behaviours	Marks
• uses t = 1 and P = 25 to find the correct value of K	1
• uses P = 0 to find the value of t	1
• states a valid prediction	1

Question 10(a) **(3 marks)**

Solution

Area of triangle $= \frac{1}{2} \times x \times x \times \sin 60^\circ = \frac{\sqrt{3}x^2}{4}$

Hence,

$A = xy + \frac{\sqrt{3}x^2}{4}$ $\qquad 3x + 2y = 10$

$= x\left(\frac{10-3x}{2}\right) + \frac{\sqrt{3}x^2}{4}$ $\qquad y = \frac{10-3x}{2}$

$ie\ A = 5x - \frac{3x^2}{2} + \frac{\sqrt{3}x^2}{4} = \frac{(\sqrt{3}-6)}{4}x^2 + 5x$

Mathematical behaviours	Marks
• determines area of triangle as an exact value	1
• states formula for total area in terms of x	1
• clearly demonstrates rearrangement of formulae to achieve required result.	1

Question 10(b) **(3 marks)**

Solution

$A = \frac{(\sqrt{3}-6)}{4}x^2 + 5x$

$\frac{dA}{dx} = 2 \times \frac{(\sqrt{3}-6)}{4}x + 5$

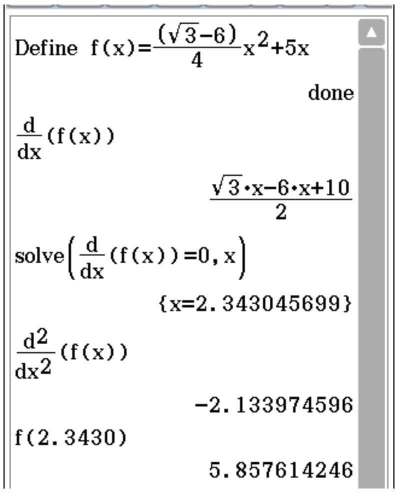
$\frac{dA}{dx} \approx -2.1340x + 5$

$\frac{dA}{dx} = 0 \Rightarrow -2.1340x + 5 = 0 \Rightarrow x \approx 2.34.$

$\frac{d^2A}{dx^2} = -2.134 < 0 \Rightarrow$ maximum

$x = 2.34 \Rightarrow A = -1.067 \times (2.34)^2 + 5 \times 2.34 \approx 5.86$

Hence the maximum area is approximately 5.86 m^2.



Mathematical behaviours	Marks
• equates $\frac{dA}{dx}$ to 0 and solves	1
• determines $\frac{d^2A}{dx^2}$ or otherwise justifies maximum	1
• calculates maximum area	1

Question 11(a) **(4 marks)**

Solution

$\frac{1}{10} + b + \frac{1}{5} + \frac{1}{5} + \frac{2}{5} = 1 \Rightarrow b = \frac{1}{10}$

$E(X) = 1 \times b + 3 \times \frac{1}{5} + 4 \times \frac{1}{5} + a \times \frac{2}{5} = 3.5$

ie $\frac{1}{10} + \frac{3}{5} + \frac{4}{5} + \frac{2a}{5} = 3.5 \Rightarrow a = 5$

Mathematical behaviours	Marks
• equates the sum of probabilities to 1	1
• evaluates b	1
• states expression for $E(X)$	1
• evaluates a	1

Question 11(b) **(5 marks)**

Solution	
(i) $\sigma^2 = \Sigma(x-\mu)^2 p(x) = (0-3.5)^2(0.1)+(1-3.5)^2(0.1)+(3-3.5)^2(0.2)+(4-3.5)^2(0.2)$ $+(5-3.5)^2(0.4) = 2.85 \Rightarrow std\ \ dev = \sqrt{2.85} \approx 1.69$ (ii) Standard deviation of $3-2X = 2\times$standard deviation of $X = 2\times 1.69 = 3.38$	
Mathematical behaviours	**Marks**
(i) • states expression to determine the variance of X • evaluates variance • evaluates standard deviation (ii) • states correct result	 1 1 1 1

Question 12(a) **(2 marks)**

Solution

h	a	$\frac{a^h - 1}{h}$
0.1	2	0.7177
0.01	2	0.6956
0.001	2	0.6934
0.0001	2	0.6932

Mathematical behaviours	Marks
• completes two rows of the table correctly	1
• completes all rows of the table correctly	1

Question 12(b) **(1 mark)**

Solution	
$\lim_{h\to 0}\frac{2^h-1}{h} = 0.693$, correct to 3 decimal places.	
Mathematical behaviours	**Marks**
• evaluates limit correctly	1

Question 12(c) **(2 marks)**

Solution	
(i) $\lim_{h\to 0}\frac{a^h-1}{h}=3\Rightarrow a\approx 20.1$ (ii) $\lim_{h\to 0}\frac{a^h-1}{h}=1\Rightarrow a=e$	
Mathematical behaviours	**Marks**
(i) • states solution (ii) • states exact solution	 1 1

Question 13(a) **(1 mark)**

Solution

x	1	2	3	4
y	2	$2\frac{1}{4}$	$3\frac{1}{9}$	$4\frac{1}{16}$

Mathematical	Marks
• states all three correct values	1

Question 13(b) **(4 marks)**

Solution	
(i) Area of lower rectangles $= 2+2\frac{1}{4}+3\frac{1}{9}$ $=\frac{265}{36}$ (ii) Area of upper rectangles $=2\frac{1}{4}+3\frac{1}{9}+4\frac{1}{16}$ $=\frac{5428}{576}=\frac{1357}{144}$	
Mathematical behaviours	**Marks**
(i) • sums the correct rectangles • deduces correct result (ii) • sums the correct rectangles • evaluates correctly	 1 1 1 1

Question 13(c) **(1 mark)**

Solution	
Estimated area $=\frac{1}{2}\left[\frac{265}{36}+\frac{5428}{576}\right]\approx 8.39$	
Mathematical behaviours	Marks
• calculates the average of the lower and upper areas	1

Question 13(d) **(1 mark)**

Solution	
Area under the curve is $\int_1^4 x+\frac{1}{x^2}\,dx=\frac{33}{4}=8.25$	
Mathematical behaviours	Marks
• states the correct answer	1

Question 14(a) **(1 mark)**

Solution	
Some people would read both digital and print. If these entries are summed those people will be counted twice.	
Mathematical behaviours	Marks
• recognises that some people will read both forms of publication	1

Question 14(b) **(2 marks)**

Solution	
$P(\text{reading print media})=\frac{6547000}{20289938}=0.322\approx 32\%$	
Mathematical behaviours	Marks
• uses correct numerator • uses correct denominator and deduces result	1 1

Question 14(c) **(3 marks)**

Solution	
$X \sim Bin(10, 0.32)$ $\mu=10\times 0.32=3.2$ $\sigma^2=10\times 0.32\times 0.68\approx 2.176$	
Mathematical behaviours	Marks
• states Binomial • calculates mean • calculates variance	1 1 1

Question 14(d) **(5 marks)**

Solution
(i) $P(X=5)\approx 0.1229$ (ii) $P(X>5)\approx 0.0637$ (iii) ${}^3C_1(0.32)(0.68)^2\times(0.32)\approx 0.1420$

Mathematical behaviours	Marks
(i)	
• states correct probability	1
(ii)	
• states appropriate probability expression	1
• calculates probability	1
(iii)	
• states correct expression for first three outcomes	1
• states fourth outcome and calculates probability	1

Question 14(e) **(3 marks)**

Solution
Let Y be the random variable denoting the number of people in the 200 who read print media. $Y\sim Bin(200, 0.32)$ $P(\text{less than 75\% do not read}) = P(\text{25\% or more do read})$ $P(Y\geq 50)=0.9874$

Mathematical behaviours	Marks
• changes parameter of distribution	1
• states appropriate probability statement	1
• evaluates	1

Question 15(a) **(2 marks)**

Solution
$\frac{d}{dx}\left[\int_0^x f(t)dt+\int_1^x t^3 f(t)dt\right]$ $=\frac{d}{dx}\int_0^x f(t)dt+\frac{d}{dx}\int_1^x t^3 f(t)dt$ $= f(x)+x^3 f(x)$

Mathematical behaviours	Marks
• applies linearity for derivatives	1
• applies the Fundamental Theorem and evaluates, stating the correct result	1

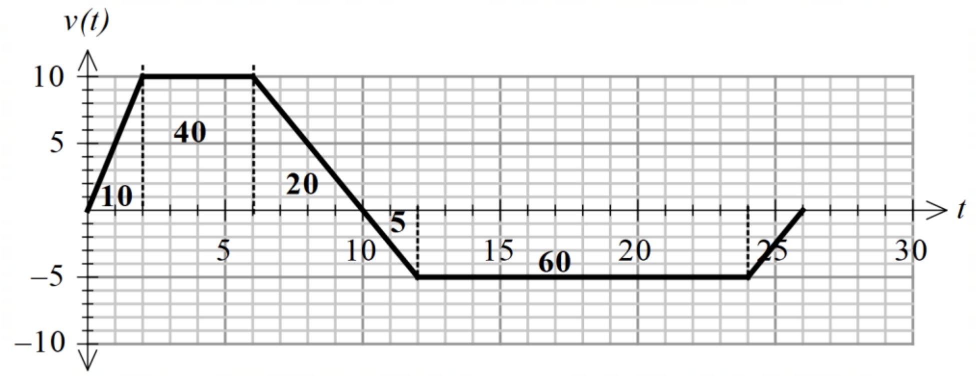
Question 15(b) **(2 marks)**

Solution	
$\int_0^x f(t)dt + \int_1^x t^3 f(t)dt = x^3 + \frac{1}{2}x^6$ $\Rightarrow \frac{d}{dx}\left[\int_0^x f(t)dt + \int_1^x t^3 f(t)dt\right] = \frac{d}{dx}\left[x^3 + \frac{1}{2}x^6\right]$ $ie\ \ f(x) + x^3 f(x) = 3x^2 + 3x^5$ $ie\ f(x)(1+x^3) = 3x^2(1+x^3)$ $ie\ \ f(x) = 3x^2$	
Mathematical behaviours	**Marks**
• differentiates both sides of the equation (or applies result from part (a)) • determines result	1 1

Question 16(a) **(1 mark)**

Solution	
$a = \frac{dv}{dt} = 0$	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 16(b) **(1 mark)**

Solution	
 $10 + 40 = 50$ Distance travelled in the first 6 seconds is 50 metres	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 16(c) **(1 mark)**

Solution	
$(10+40+20) - 5 = 65$ Displacement after 12 seconds is 65 m	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 16(d) **(1 mark)**

Solution	
Distance travelled after 12 seconds is 10+40+20+5 =75 m	
Mathematical behaviours	Marks
• states correct answer	1

Question 16(e) **(1 mark)**

Solution	
At $t = 11$ both the velocity and the acceleration are negative hence the particle is speeding up.	
Mathematical behaviours	Marks
• states the particle is speeding up	1

Question 17(a) **(2 marks)**

Solution

y	0	1
$\mathrm{P}(Y=y)$	$\frac{6}{10}$	$\frac{4}{10}$

Mathematical behaviours	Mark
• completes first probability correctly	1
• completes second probability correctly	1

Question 17(b) **(2 marks)**

Solution	
It is a Bernoulli distribution with mean = $\frac{4}{10}$.	
Mathematical behaviours	Marks
• states the distribution name • states the mean	1 1

Question 17(c) **(2 marks)**

Solution	
$X = 0,1$ or 2	
Mathematical behaviours	Marks
• states all values	1

Question 17(d) **(1 mark)**

Solution	
$P(X = 0) = P$ (not prime and not prime) $= \frac{6}{10} \times \frac{6}{10} = \frac{36}{100}$	
Mathematical behaviours	Mark
• calculates probability	1

Question 17(e) **(3 marks)**

Solution	
$P(X=1)=\frac{4}{10}\times\frac{6}{10}+\frac{6}{10}\times\frac{4}{10}=\frac{48}{100}$ $\quad$ $P(X=2)=\frac{4}{10}\times\frac{4}{10}=\frac{16}{100}$ Hence X=1 is the most likely result.	
Mathematical behaviours	**Mark**
• calculates $P(X=1)$ • calculates $P(X=2)$ • states correct conclusion	1 1 1

Question 17(f) **(3 marks)**

Solution	
Let the random variable F represent the operator's financial position for each game.	
Mathematical behaviours	**Marks**
• determines expected value for 1 game • calculates gain for the day • states final outcome, with unit and explains	1 1 1

f	-2	5
$P(F=f)$	$\frac{52}{100}$	$\frac{48}{100}$

$$E(F)=-2\times\frac{52}{100}+5\times\frac{48}{100}=1.36$$

Hence the operator will expect to make a profit of $1.36 per game in the long term. With 500 contestants he will expect to make $500\times\$1.36=\$680$

Question 17(g) **(2 marks)**

Solution	
Let k be the charge to play the game.	
Mathematical behaviours	**Marks**
• constructs equation for expected value • solves equation to determine k	1 1

f	(k-7)	k
$P(F=f)$	$\frac{52}{100}$	$\frac{48}{100}$

$$E(F)=(k-7)\times\frac{52}{100}+k\times\frac{48}{100}=0\Rightarrow k-\frac{364}{100}=0\Rightarrow k=3.64.$$

Hence the operator would need to charge $3.64.

Question 18(a) **(6 marks)**

Solution	
(i) Since the maximum and minimum values are 14.5 and 9.5 $a+b=14.5$ and $a-b=9.5 \Rightarrow a=12$ and $b=2.5$. or mean line $=\dfrac{14.5+9.5}{2} \Rightarrow a=12$ and amplitude $=b=\dfrac{14.5-9.5}{2}=2.5$ Since the period of the oscillation is 12, $c=\dfrac{2\pi}{12}\approx 0.5236$. (ii) $S=12+2.5\cos(0.5236t+d)$, $\dfrac{dS}{dt}=-2.5\times(0.5236)\sin(0.5236t+d)$ $\dfrac{dS}{dt}=0 \Rightarrow \sin(0.5236t+d)=0$ Maximum at $t = 11.7$ $\Rightarrow 0.5236\times 11.7+d=2\pi$ $d\approx 0.1571$	
Mathematical behaviours	**Marks**
(i)	
• explains exactly one of a and b values	1
• explains both a and b values	1
• identifies the period to explain the value of c	1
(ii)	
• differentiates correctly	1
• equates to *0* and equates angle to 2π	1
• solves equation to determine d	1

Question 18(b) **(1 mark)**

Solution	
On April 30th, $t=4$ $S=12+2.5\cos(0.5236\times 4+0.1571)\approx 10.4$ hours So we can expect 10.4 hours of sunlight on April 30th.	
Mathematical behaviours	**Marks**
• states correct answer	1

Question 18(c) **(2 marks)**

Solution

$$\int_4^6 12 + 2.5\cos(0.5236t + 0.1571)\,dt \approx 19.54$$

So average $= \dfrac{19.54}{2} \approx 9.77.$

So the average daily amount in May and June is 9.77 hours

Mathematical behaviours	Marks
• uses correct integral	1
• states solution	1

Question 18(d) **(3 marks)**

Solution

$$S = 12 + 2.5\cos(0.5236t + 0.1571)$$

$$\frac{dS}{dt} = -2.5(0.5236)\sin(0.5236t + 0.1571)$$

Hence the maximum value of $\dfrac{dS}{dt}$ is $2.5 \times 0.5236 \approx 1.31$ hours per month

Using the increments formula $\delta S \approx \dfrac{dS}{dt} \times \delta t$, the maximum change in successive days is approximately $1.31 \times \dfrac{1}{30}$ hours.

i.e. 2.62 minutes (or 3 minutes to the nearest minute).

Mathematical behaviours	Marks
• identifies maximum value of $\frac{dS}{dt}$	1
• substitutes into increments formula correctly	1
• states answer to the nearest minute	1

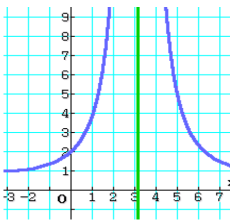
Question 19(a) **(3 marks)**

Solution

$A(m) = \int_0^m f(x)\,dx$ represents the area bounded by the curve,

the line $x = 0$ and the line $x = m$.

The function is undefined at $m = \pi$ since $\left(1 - \sin\dfrac{x}{2} = 0\right)$ hence the area cannot be calculated



Mathematical behaviours	Marks
• relates the integral to an area under the curve	1
• states that $m = \pi$ since f is undefined at $x = \pi$	1
• concludes that the area cannot be calculated – it id not bounded	1

Question 19(b) **(5 marks)**

Solution

$$f(x)=g(x)\Rightarrow \frac{2}{1-\sin\left(\frac{x}{2}\right)}=-(x-2)^2+6$$

$$\Rightarrow x\approx 1.4038193$$

$$\text{Area}=\left[\int_0^{1.4038193}\left(-(x-2)^2+6\right)-\left(\frac{2}{1-\sin\left(\frac{x}{2}\right)}\right)\right]dx$$

$$+\int_{1.4038193}^{2}\left[\left(\frac{2}{1-\sin\left(\frac{x}{2}\right)}\right)-\left(-(x-2)^2+6\right)\right]dx$$

$$=1.2064+1.5059$$

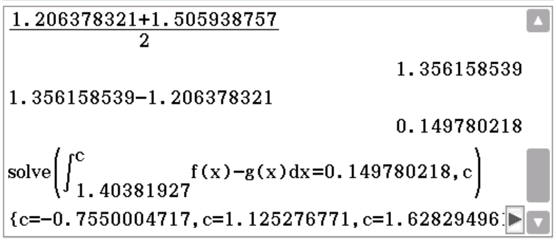
$$=2.7123\approx 2.71$$

Define $f(x)=\frac{2}{1-\sin\frac{x}{2}}$
done
Define g(x)=-(x-2)^2+6
done
solve(f(x)=g(x),x)
{x=0,x=1.40381927}
$\int_0^{1.40381927}(g(x)-f(x))dx$
1.206378321
$\int_{1.40381927}^{2}(f(x)-g(x))dx$
1.505938757
1.206378321+1.505938757
2.712317078

Mathematical behaviours	Marks
• determines point of intersection of f and g	1
• states appropriate integral to determine first area	1
• states appropriate integral to determine second area	1
• evaluates one integral correctly	1
• determines correct result to two decimal places	1

Question 19(c) **(3 marks)**

Solution
$\dfrac{1.2064+1.5059}{2}=1.3562$ Hence $1.4038193 < x < 2$ $1.3562-1.2064=0.1498$ $\displaystyle\int_{1.4038193}^{c}\left[\left(\frac{2}{1-\sin\left(\frac{x}{2}\right)}\right)-\left(-(x-2)^2+6\right)\right]dx=0.1498 \Rightarrow x=-0.7550,1.1253,1.6282$ Hence $c\approx 1.63$



Mathematical behaviours	Marks
• determines average of two areas	1
• states appropriate equation to be solved involving integral	1
• solves equation and concludes solution	1